

Quantization of Space-time Noncommutative Theory¹

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Abstract

General aspects of the quantization of field theories non-local in time are discussed. The path integral on the basis of Schwinger's action principle and the Bjorken-Johnson-Low prescription, which helps to recover the canonical structure from the results of the path integral, are used as the main machinery. A modified time ordering operation which formally restores unitarity in field theories non-local in time is analyzed in detail. It is shown that the perturbative unitarity and the positive energy condition, in the sense that only the positive energy flows in the positive time direction for any fixed time-slice in space-time, are not simultaneously satisfied for theories non-local in time such as space-time noncommutative theory.

1 Introduction

The quantization of field theories non-local in time is problematic. First of all, no canonical formulation of such theories is known since a sensible definition of canonical momenta is not known, and the violation of unitarity appears to be the general aspects of such theories[1]. This issue has re-appeared in the recent study of space-time noncommutative theories[2, 3, 4]. An interesting new development in this subject is that a suitable modified definition of time-ordering operation restores the perturbative unitarity in space-time noncommutative theory [5, 6, 7], though its consistency with other basic postulates in conventional quantized field theory remains to be clarified.

We discuss the issues related to unitarity and the modified time ordering prescription on the basis of a recent paper[8], which analyzed these issues by using path integral formulation and the Bjorken-Johnson-Low (BJL)[9] prescription.

2 Quantization of higher derivative theory

We present a path integral formulation of higher derivative theory and then show how to recover the canonical structure from the path integral. For simplicity, we study the theory defined by

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) + \lambda \frac{1}{2} \phi(x) \square^2 \phi(x) \quad (2.1)$$

where² $\square = \partial_\mu \partial^\mu$ and λ is a real constant. The canonical formulation of higher derivative theory such as the present one has been analyzed in [10], for example.

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²Our metric convention is $g_{\mu\nu} = (1, -1, -1, -1)$.

We instead start with Schwinger's action principle and consider the Lagrangian with a source function $J(x)$

$$\mathcal{L}_J = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + \lambda\frac{1}{2}\phi(x)\square^2\phi(x) + J(x)\phi(x). \quad (2.2)$$

The Schwinger's action principle starts with the equation of motion

$$\begin{aligned} & \langle +\infty | -\square\hat{\phi}(x) + \lambda\square^2\hat{\phi}(x) + J(x) | -\infty \rangle_J \\ &= \left\{ -\square\frac{\delta}{i\delta J(x)} + \lambda\square^2\frac{\delta}{i\delta J(x)} + J(x) \right\} \langle +\infty | -\infty \rangle_J = 0. \end{aligned} \quad (2.3)$$

We here assume the existence of a formally quantized field $\hat{\phi}(x)$, though its detailed properties are not specified yet, and the asymptotic states $|\pm\infty\rangle_J$ in the presence of a source function $J(x)$ localized in space-time. The path integral is then defined as a formal solution of the above functional equation

$$\langle +\infty | -\infty \rangle_J = \int \mathcal{D}\phi \exp\{i \int d^4x \mathcal{L}_J\}. \quad (2.4)$$

We now define the Green's function by

$$\begin{aligned} \langle +\infty | T^*\hat{\phi}(x)\hat{\phi}(y) | -\infty \rangle &= \frac{\delta}{i\delta J(x)} \frac{\delta}{i\delta J(y)} \langle +\infty | -\infty \rangle_J|_{J=0} \\ &= \frac{1}{i} \frac{1}{\square - i\epsilon - \lambda\square^2} \delta(x-y). \end{aligned} \quad (2.5)$$

This Green's function contains all the information about the quantized field.

The BJJ prescription states that we can replace the covariant T^* product by the conventional T product when

$$\lim_{k^0 \rightarrow i\infty} \int d^4x e^{ik(x-y)} \langle +\infty | T^*\hat{\phi}(x)\hat{\phi}(y) | -\infty \rangle = \lim_{k^0 \rightarrow i\infty} \frac{i}{k^2 + i\epsilon + \lambda(k^2)^2} = 0. \quad (2.6)$$

Thus we have

$$\int d^4x e^{ik(x-y)} \langle +\infty | T\hat{\phi}(x)\hat{\phi}(y) | -\infty \rangle = \frac{i}{k^2 + i\epsilon + \lambda(k^2)^2}. \quad (2.7)$$

By multiplying a suitable powers of the momentum variable k_μ , we can recover the canonical commutation relations. For example,

$$\begin{aligned} & k_\mu \int d^4x e^{ik(x-y)} \langle +\infty | T\hat{\phi}(x)\hat{\phi}(y) | -\infty \rangle \\ &= \int d^4x (-i\partial_\mu^x e^{ik(x-y)}) \langle +\infty | T\hat{\phi}(x)\hat{\phi}(y) | -\infty \rangle \\ &= \int d^4x e^{ik(x-y)} \{ \langle +\infty | T i\partial_\mu^x \hat{\phi}(x)\hat{\phi}(y) | -\infty \rangle \\ &\quad + i\delta(x^0 - y^0) \langle +\infty | [\hat{\phi}(x), \hat{\phi}(y)] | -\infty \rangle \} \\ &= \frac{ik_\mu}{k^2 + i\epsilon + \lambda(k^2)^2}. \end{aligned} \quad (2.8)$$

An analysis of this relation in the limit $k_0 \rightarrow i\infty$ gives

$$\delta(x^0 - y^0)[\hat{\phi}(x), \hat{\phi}(y)] = 0, \\ \int d^4x e^{ik(x-y)} \{ \langle +\infty | T i \partial_\mu^x \hat{\phi}(x) \hat{\phi}(y) | -\infty \rangle = \frac{ik_\mu}{k^2 + i\epsilon + \lambda(k^2)^2}. \quad (2.9)$$

Note that the limit $k_0 \rightarrow i\infty$ of the Fourier transform of a T product such as $\langle +\infty | T i \partial_\mu^x \hat{\phi}(x) \hat{\phi}(y) | -\infty \rangle$ vanishes *by definition*.

By repeating the procedure with (2.9), we obtain

$$\begin{aligned} \delta(x^0 - y^0)[\hat{\phi}(x), \hat{\phi}(y)] &= 0, \\ \delta(x^0 - y^0)[\partial_0 \hat{\phi}(x), \hat{\phi}(y)] &= 0, \\ \delta(x^0 - y^0)[\partial_0^2 \hat{\phi}(x), \hat{\phi}(y)] &= 0, \\ \delta(x^0 - y^0)[\partial_0^3 \hat{\phi}(x), \hat{\phi}(y)] &= \frac{i}{\lambda} \delta^{(4)}(x - y), \\ \delta(x^0 - y^0)[\partial_0 \hat{\phi}(x), \partial_0 \hat{\phi}(y)] &= 0, \\ \delta(x^0 - y^0)[\partial_0^2 \hat{\phi}(x), \partial_0 \hat{\phi}(y)] &= -\frac{i}{\lambda} \delta^{(4)}(x - y). \end{aligned} \quad (2.10)$$

The general rule is that the commutator

$$[\hat{\phi}^{(m)}(x), \hat{\phi}^{(l)}(y)] \delta(x^0 - y^0) \neq 0 \quad (2.11)$$

where $m + l = n - 1$ for a theory with the n -th time derivative. Here $\hat{\phi}^{(l)}(x)$ stands for the l -th time derivative of $\hat{\phi}(x)$, $\hat{\phi}^{(l)}(x) = \frac{\partial^l}{\partial(x^0)^l} \hat{\phi}(x)$. We thus derive all the canonical commutation relations (2.10) from the path integral defined by the Schwinger's action principle and the T^* product, and those commutation relations naturally agree with the relations derived by a canonical formulation of the higher derivative theory[10].

3 Quantization of a theory non-local in time

We examine a non-local theory defined by

$$\mathcal{L}_J = -\frac{1}{2} \phi(x) \square [\phi(x + \xi) + \phi(x - \xi)] + J(x) \phi(x). \quad (3.1)$$

A formal integration of the Schwinger's action principle gives a path integral

$$\langle +\infty | -\infty \rangle_J = \int \mathcal{D}\phi \exp\{i \int d^4x \mathcal{L}_J\}, \quad (3.2)$$

which in turn leads to the correlation function

$$\int d^4x e^{ik(x-y)} \langle T^* \hat{\phi}(x) \hat{\phi}(y) \rangle = \frac{i}{(k^2 + i\epsilon)[e^{ik\xi} + e^{-ik\xi}]}. \quad (3.3)$$

For a time-like vector ξ , which may be chosen as $(\xi^0, 0, 0, 0)$, the right-hand side of this expression multiplied by any power of k_0 goes to zero for k_0 along the imaginary axis in

the complex k_0 plane. Thus the application of BJL prescription leads to (for any pair of non-negative integers n and m)

$$[\hat{\phi}^{(n)}(x), \hat{\phi}^{(m)}(y)]\delta(x^0 - y^0) = 0 \quad (3.4)$$

where $\hat{\phi}^{(n)}(x)$ stands for the n -th time derivative of $\hat{\phi}(x)$ as in (2.11). This relation is consistent with the $N \rightarrow \infty$ limit of a higher derivative theory obtained by a truncation of the power series expansion of $e^{\pm i\xi\hat{p}}$ at the N -th power in the starting Lagrangian (3.1).

We next analyze a theory which contains a non-local interaction

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{m^2}{2}\phi(x)\phi(x) \\ & - \frac{g}{2}[\phi(x+\xi)\phi(x)\phi(x-\xi) + \phi(x-\xi)\phi(x)\phi(x+\xi)] + \phi(x)J(x) \end{aligned} \quad (3.5)$$

where ξ^μ is a constant four-vector. This theory is not Lorentz invariant because of the constant vector ξ^μ . When one chooses ξ^μ to be $\xi^\mu = (\xi^0, 0, 0, 0)$ with $\xi^0 > 0$, the quantization of the above theory is analogous to that of space-time noncommutative theory.

One may study the path integral quantization without specifying the precise quantization condition of field variables. One may thus define a path integral by means of Schwinger's action principle and a suitable ansatz of asymptotic conditions as in (3.2)

$$\langle +\infty | -\infty \rangle_J = \int \mathcal{D}\phi \exp[i \int d^4x \mathcal{L}_J]. \quad (3.6)$$

One may then define a formal expansion in powers of the coupling constant g . We study one-loop diagrams in a formal perturbative expansion in powers of the coupling constant g by starting with a tentative ansatz of quantization

$$\langle T^* \hat{\phi}(x) \hat{\phi}(y) \rangle = \frac{-i}{\square + m^2 - i\epsilon} \delta(x - y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{i}{k^2 - m^2 + i\epsilon} \quad (3.7)$$

which is equivalent to a canonical quantization of free theory. One-loop self-energy diagrams contain the contribution

$$\frac{(-ig)^2}{2} \int d^4x d^4y \phi(x) \phi(y) \langle T^* \phi(x+\xi) \phi(y-\xi) \rangle \langle T^* \phi(x-\xi) \phi(y-2\xi) \rangle \quad (3.8)$$

which contains the finite non-local terms separated up to the order of $\sim 3\xi$.

This term in (3.8) gives rise to

$$\begin{aligned} & g^2 i \Sigma(k, \xi) \\ = & \frac{ig^2}{2(4\pi)^2} \int_0^1 dx [e^{i(2-x)\xi p} + e^{-i(2-x)\xi p}] \int_0^\infty \frac{d\alpha}{\alpha} e^{i\alpha[x(1-x)p^2 - m^2 + i\epsilon] - i\frac{\xi^2}{4\alpha}}. \end{aligned} \quad (3.9)$$

Following the conventional approach, we define the integral for a Euclidean momentum p_μ , for which $p^2 < 0$. In this case, one can deform the integration contour in the variable α along the negative real axis and obtain

$$\int_0^\infty \frac{d\alpha}{\alpha} e^{i\alpha[x(1-x)p^2 - m^2 + i\epsilon] - i\frac{\xi^2}{4\alpha}} = -i\pi H_0^{(2)}(-i\sqrt{(-\xi^2)[-x(1-x)p^2 + m^2 - i\epsilon]}) \quad (3.10)$$

for $\xi^2 < 0$. Here $H_0^{(2)}(z)$ stands for the Hankel function which has an asymptotic expansion for $|z| \rightarrow \infty$

$$H_0^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{-i(z - \frac{\pi}{4})} \quad (3.11)$$

for $-2\pi < \arg z < \pi$.

We thus find that for $p_0 \rightarrow i\infty$

$$\int_0^\infty \frac{d\alpha}{\alpha} e^{i\alpha[x(1-x)p^2 - m^2 + i\epsilon] - i\frac{\xi^2}{4\alpha}} \sim -\pi \sqrt{\frac{2}{\pi z}} e^{-z} \quad (3.12)$$

with $z = \sqrt{(-\xi^2)[-x(1-x)p^2 + m^2 - i\epsilon]}$ for a space-like ξ , $\xi^2 < 0$. On the other hand, we have a damping oscillatory behavior for $p_0 \rightarrow i\infty$,

$$\int_0^\infty \frac{d\alpha}{\alpha} e^{i\alpha[x(1-x)p^2 - m^2 + i\epsilon] - i\frac{\xi^2}{4\alpha}} \sim -i\pi \sqrt{\frac{2}{\pi z}} e^{(-iz + i\frac{\pi}{4})} \quad (3.13)$$

with $z = \sqrt{(\xi^2)[-x(1-x)p^2 + m^2 - i\epsilon]}$ for a time-like ξ , $\xi^2 > 0$, which is defined by an analytic continuation.

When one writes the (complete) connected two-point correlation function with one-loop corrections as

$$\langle T^* \hat{\phi}(x) \hat{\phi}(y) \rangle_{ren} = \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{p^2 + g^2 \Sigma(p, \xi) - m_r^2 + i\epsilon}, \quad (3.14)$$

the two-point function generally contains the non-local term in $g^2 \Sigma(p, \xi)$. When one applies the B JL prescription to the two-point correlation function, for the time-like ξ for which we may take $\xi = (\xi^0, \vec{0})$, $g^2 \Sigma(p, \xi)$ diverges exponentially for $p_0 \rightarrow i\infty$. This arises from the behavior of the factor $[e^{i(2-x)\xi p} + e^{-i(2-x)\xi p}]$ in (3.9) for $p_0 \rightarrow i\infty$ and $\xi = (\xi^0, \vec{0})$, which dominates the damping oscillatory behavior (3.13)³. The canonical structure specified by the B JL analysis is thus completely modified by the one-loop effects of the interaction non-local in time. After one-loop corrections, we essentially have the same result (3.3) as for the non-local theory (3.1). The naive ansatz of the two-point correlation function at the starting point of perturbation theory (3.7) is not justified.

Nevertheless, it is instructive to examine the formal perturbative unitarity of an S-matrix defined for the theory non-local in time. One may start with a naive Hamiltonian

$$\begin{aligned} \mathcal{H} = & \frac{1}{2} \Pi^2(x) + \frac{1}{2} \vec{\nabla} \phi(x) \vec{\nabla} \phi(x) + \frac{1}{2} m^2 \phi^2(x) \\ & + \frac{g}{2} [\phi(x - \xi) \phi(x) \phi(x + \xi) + \phi(x + \xi) \phi(x) \phi(x - \xi)] \end{aligned} \quad (3.15)$$

where $x^\mu = (0, \vec{x})$ and $\xi^\mu = (\xi^0, \vec{0})$, and $\Pi(x) = \frac{\partial}{\partial x^0} \phi(x)$ is a naive canonical momentum conjugate to $\phi(x)$. This Hamiltonian is formally hermitian, $\mathcal{H}^\dagger = \mathcal{H}$, but \mathcal{H} is not local

³The non-vanishing imaginary part of $g^2 \Sigma(p, \xi)$ in (3.13) for the Euclidean momentum given by $p_0 \rightarrow i\infty$ is associated with the violation of unitarity in the present theory non-local in time [2, 3, 4].

in the time coordinate and does not generate time development in the conventional sense for the small time interval $\Delta t < \xi^0$.

One may then observe that

$$S(t_+, t_-) = e^{i\hat{H}_0 t_+} e^{-i\hat{H}(t_+ - t_-)} e^{-i\hat{H}_0 t_-} \quad (3.16)$$

for \mathcal{H} in (3.6) with H_0 the free Hamiltonian is unitary, $S(t_+, t_-)^\dagger S(t_+, t_-) = S(t_+, t_-) S(t_+, t_-)^\dagger = 1$. The formal power series expansion in the coupling constant⁴

$$S(t_+, t_-) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_-}^{t_+} dt_1 \dots \int_{t_-}^{t_+} dt_n T_\star(\hat{H}_I(t_1) \dots \hat{H}_I(t_n)) \quad (3.17)$$

with a hermitian

$$\hat{H}_I(t) = e^{i\hat{H}_0 t} \int d^3x \frac{g}{2} [\phi(-\xi, \vec{x}) \phi(0, \vec{x}) \phi(\xi, \vec{x}) + \phi(\xi, \vec{x}) \phi(0, \vec{x}) \phi(-\xi, \vec{x})] e^{-i\hat{H}_0 t} \quad (3.18)$$

thus defines a unitary S-matrix for $t_- \rightarrow -\infty, t_+ \rightarrow +\infty$. This definition of a unitary operator corresponds to the definition of a unitary S-matrix for space-time noncommutative theory proposed in [5, 6].

It is important to recognize that the time-ordering in the present context is defined with respect to the time variable appearing in $\hat{H}_I(t)$; if one performs a time-ordering with respect to the time variable appearing in each field variable $\phi(x)$, one generally obtains different results due to the non-local structure of the interaction term in time. Since the operator \hat{S} defined above is manifestly unitary, the non-unitary result in the conventional Feynman rules, which are based on the time-ordering of each operator $\phi(x)$, arises from this difference of time ordering.

When one defines

$$\begin{aligned} A_1 &= \int_{-\infty}^{\infty} dt \hat{H}_I(t), \\ A_2 &= \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt_1 dt_2 T_\star \hat{H}_I(t_1) \hat{H}_I(t_2) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt_1 dt_2 \theta(t_1 - t_2) \hat{H}_I(t_1) \hat{H}_I(t_2) \end{aligned} \quad (3.19)$$

the unitarity relation of the above S-matrix in the second order of the coupling constant requires(see, for example, [6])

$$A_2 + A_2^\dagger = A_1^\dagger A_1 = A_1^2. \quad (3.20)$$

To be explicit

$$\begin{aligned} A_2 + A_2^\dagger &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt_1 dt_2 \theta(t_1 - t_2) \{ \hat{H}_I(t_1) \hat{H}_I(t_2) + \hat{H}_I(t_2) \hat{H}_I(t_1) \} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt_1 dt_2 \{ \theta(t_1 - t_2) \hat{H}_I(t_1) \hat{H}_I(t_2) + \theta(t_2 - t_1) \hat{H}_I(t_1) \hat{H}_I(t_2) \} \\ &= \int_{-\infty}^{\infty} dt_1 \hat{H}_I(t_1) \int_{-\infty}^{\infty} dt_2 \hat{H}_I(t_2) = A_1^2 \end{aligned} \quad (3.21)$$

⁴We use the notation T_\star for the time ordering in non-local theory, whereas T or T^\star is used for the conventional time ordering with respect to the time variable of *each* field variable $\phi(x)$.

by noting $\theta(t_1 - t_2) + \theta(t_2 - t_1) = 1$, as required by the unitarity relation.

In contrast, if one uses the conventional time ordering one has

$$\begin{aligned} A_2 &= \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt_1 dt_2 T^* \hat{H}_I(t_1) \hat{H}_I(t_2) \\ &\neq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt_1 dt_2 \theta(t_1 - t_2) \hat{H}_I(t_1) \hat{H}_I(t_2) \end{aligned} \quad (3.22)$$

since the time ordering by T^* is defined with respect to the time variable of each field $\phi(t, \vec{x})$, and thus the unitarity of the conventional operator

$$\hat{S} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} dt_1 \dots \int_{-\infty}^{+\infty} dt_n T^*(\hat{H}_I(t_1) \dots \hat{H}_I(t_n)) \quad (3.23)$$

is not satisfied for the non-local $\hat{H}_I(t)$ in general. Note that the perturbative expansion with the T^* product is directly defined by the path integral without recourse to the expression such as (3.15).

On the other hand, the positive energy condition, which is ensured by the Feynman propagator, is not obvious for the propagator defined by the modified time ordering T_* . To be specific, we have the following correlation function in the Wick-type reduction of the S-matrix in (3.23)

$$\begin{aligned} &\langle 0 | T \phi(x - \xi) \phi(y + \xi) | 0 \rangle \\ &= \int \frac{d^4 k}{(2\pi)^4} \exp[-ik((x - \xi) - (y + \xi))] \frac{i}{k_\mu k^\mu - m^2 + i\epsilon} \\ &= \theta((x - \xi)^0 - (y + \xi)^0) \int \frac{d^3 k}{(2\pi)^3 2\omega} \exp[-i\omega((x - \xi)^0 - (y + \xi)^0) + i\vec{k}(\vec{x} - \vec{y})] \\ &\quad + \theta((y + \xi)^0 - (x - \xi)^0) \int \frac{d^3 k}{(2\pi)^3 2\omega} \exp[-i\omega((y + \xi)^0 - (x - \xi)^0) + i\vec{k}(\vec{y} - \vec{x})] \end{aligned} \quad (3.24)$$

with $\omega = \sqrt{\vec{k}^2 + m^2}$ for the conventional Feynman prescription with $m^2 - i\epsilon$, which ensures that the positive frequency components propagate in the forward time direction and the negative frequency components propagate in the backward time direction and thus the positive energy flows always in the forward time direction. The path integral with respect to the field variable $\phi(x)$ gives this time ordering or the T^* product.

In comparison, the non-local prescription (3.17) gives the following correlation function for the quantized free field in the Wick-type reduction

$$\begin{aligned} &\langle 0 | T_* \phi(x - \xi) \phi(y + \xi) | 0 \rangle \\ &= \theta(x^0 - y^0) \int \frac{d^3 k}{(2\pi)^3 2\omega} \exp[-i\omega((x - \xi)^0 - (y + \xi)^0) + i\vec{k}(\vec{x} - \vec{y})] \\ &\quad + \theta(y^0 - x^0) \int \frac{d^3 k}{(2\pi)^3 2\omega} \exp[-i\omega((y + \xi)^0 - (x - \xi)^0) + i\vec{k}(\vec{y} - \vec{x})] \end{aligned} \quad (3.25)$$

where the time-ordering step function $\theta(x^0 - y^0)$, for example, and the signature of the time variable $(x - \xi)^0 - (y + \xi)^0$ appearing in the exponential are not correlated, and it

allows the negative energy to propagate in the forward time direction also. This result is not reproduced by the Feynman's $m^2 - i\epsilon$ prescription. When one considers an arbitrary fixed time-slice in 4-dimensional space-time, the condition that all the particles crossing the time-slice carry the positive energy in the forward time direction, which is regarded as the positive energy condition in the path integral formulation[11] (or in perturbation theory in general), is not satisfied. This positive energy condition is crucial in the analysis of spin-statistics theorem[11, 12], for example.

We thus summarize the analysis of this section as follows: The naive canonical quantization in a perturbative sense is not justified in the present theory non-local in time when one incorporates the higher order corrections. The unitarity of the (formal) perturbative S-matrix is ensured if one adopts the T_\star product, but the positive energy condition is not satisfied by this prescription. On the other hand, the unitarity of the S matrix is spoiled if one adopts the conventional T or T^\star product which is defined by the path integral, though the positive energy condition and a smooth Wick rotation are ensured.

4 Space-time noncommutative theory

We study the simplest noncommutative theory defined by

$$\begin{aligned}\mathcal{L}_J = & \frac{1}{2}\partial_\mu\phi(x)\star\partial^\mu\phi(x) - \frac{m^2}{2}\phi(x)\star\phi(x) \\ & - \frac{g}{3!}\phi(x)\star\phi(x)\star\phi(x) + \phi(x)\star J(x)\end{aligned}\quad (4.1)$$

where the \star product is defined by the so-called Moyal product

$$\phi(x)\star\phi(x) = e^{\frac{i}{2}\xi\partial_\mu^x\theta^{\mu\nu}\partial_\nu^y}\phi(x)\phi(y)|_{y=x} = e^{\frac{i}{2}\xi\partial_x\wedge\partial_y}\phi(x)\phi(y)|_{y=x}.\quad (4.2)$$

The real positive parameter ξ stands for the deformation parameter, and the antisymmetric parameter $\theta^{\mu\nu} = -\theta^{\nu\mu}$ corresponds to $i\xi\theta^{\mu\nu} = [\hat{x}^\mu, \hat{x}^\nu]$; since this theory is not Lorentz covariant we consider the case $\theta^{0i} = -\theta^{i0} \neq 0$ for a suitable i but all others $\theta^{\mu\nu} = 0$ in the following. The path integral is then defined as a formal integral of the Schwinger's action principle

$$\langle +\infty | -\infty \rangle_J = \int \mathcal{D}\phi \exp[i \int d^4x \mathcal{L}_J].\quad (4.3)$$

It has been argued [13] that the present theory is renormalizable in the formal perturbative expansion in powers of the coupling constant g starting with

$$\langle T^\star \hat{\phi}(x) \hat{\phi}(y) \rangle = \frac{-i}{\square + m^2 - i\epsilon} \delta(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{i}{k^2 - m^2 + i\epsilon} \quad (4.4)$$

which is equivalent to a canonical quantization of free theory. The one-loop self-energy is given by

$$g^2 i \Sigma(p, \xi)$$

$$\begin{aligned}
&= \frac{-g^2}{2} \int \frac{d^4 k}{(2\pi)^4} \cos^2\left(\frac{\xi}{2} p \wedge k\right) \frac{i}{((p-k)^2 - m^2 + i\epsilon)} \frac{i}{(k^2 - m^2 + i\epsilon)} \\
&= \frac{g^2}{4} \int \frac{d^4 k}{(2\pi)^4} \frac{1 + \cos(\xi p \wedge k)}{((p-k)^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)}
\end{aligned} \tag{4.5}$$

Since the term without the factor $\cos(\xi p \wedge k)$ is identical to the conventional theory, we concentrate on the term with $\cos(\xi p \wedge k)$

$$\begin{aligned}
&\frac{-g^2}{8} \int \frac{d^4 k}{(2\pi)^4} [e^{i\xi p \wedge k} + e^{-i\xi p \wedge k}] \int_0^\infty \alpha d\alpha \int_0^1 dx e^{i\alpha[k^2 + x(1-x)p^2 - m^2 + i\epsilon]} \\
&= \frac{ig^2}{4(4\pi)^2} \int_0^\infty \frac{d\alpha}{\alpha} \int_0^1 dx e^{i\alpha[x(1-x)p^2 - m^2 + i\epsilon] - i\frac{\xi^2 \tilde{p}^2}{4\alpha}}
\end{aligned} \tag{4.6}$$

where

$$\tilde{p}^\mu = \theta^{\mu\nu} p_\nu. \tag{4.7}$$

See also (3.10). For space-time noncommutative theory with $\theta^{01} \neq 0$, for example, $\tilde{p}^2 \sim p_1^2 - p_0^2$, and for space-space noncommutative theory with $\theta^{23} \neq 0$, for example, $\tilde{p}^2 \sim -p_2^2 - p_3^2$. We thus obtain by using the result in (3.10)

$$\begin{aligned}
&g^2 i \Sigma(p, \xi)_{non-planar} \\
&= \frac{\pi g^2}{4(4\pi)^2} H_0^{(2)}(-i\sqrt{(-\xi^2 \tilde{p}^2)[-x(1-x)p^2 + m^2 - i\epsilon]})
\end{aligned} \tag{4.8}$$

for $\xi^2 \tilde{p}^2 < 0$, namely, for space-space noncommutative theory. For space-time noncommutative theory, for which $\xi^2 \tilde{p}^2$ can be positive as well as negative, one defines the amplitude by an analytic continuation.

As for the consistency of the naive quantization (4.4), it is important to analyze the self-energy correction in

$$\langle T^* \hat{\phi}(x) \hat{\phi}(y) \rangle_{ren} = \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{p^2 + g^2 \Sigma(p, \xi) - m_r^2}. \tag{4.9}$$

If $\Sigma(p, \xi)$ contains a non-local exponential factor, the naive quantization is not justified in the framework of the BJL prescription as we explained for the simple theory non-local in time in the previous section. By using the asymptotic expansion in (3.11), it is shown that the naive quantization (4.4), either in space-space or in space-time noncommutative theory, is not modified by the one-loop corrections in the framework of BJL prescription. This is in sharp contrast to the simple theory non-local in time analyzed in the previous section. This difference arises from the fact that $p_\mu \theta^{\mu\nu} p_\nu = 0$ and thus the two-point function, which depends on the single momentum p_μ , does not contain an extra exponential factor in the present space-time noncommutative theory. Although our analysis does not justify the naive quantization to the non-perturbative accuracy, it provides a basis of the formal perturbative expansion in the present model[14].

As for the analysis of unitarity, the argument is almost identical to that for the simple non-local theory in the previous section. The one-loop self-energy amplitude in the path integral formulation (4.6) is obtained from (3.9) by replacing

$$\xi^2 \rightarrow \xi^2 \tilde{p}^2 \tag{4.10}$$

except for the exponential prefactor. The amplitude for the Euclidean momentum exhibits oscillatory behavior with a non-vanishing imaginary part for the space-time noncommutative theory; this indicates the violation of unitarity [2, 3, 4] in the path integral formulation with the conventional time-ordering. On the other hand, the amplitude for the space-space noncommutative theory is real and exponentially damping for the Euclidean momentum, and thus no violation of unitarity in the conventional time-ordering. See also (3.12) and (3.13).

On the other hand, the analysis of unitarity on the basis of modified time-ordering [5, 6] starts with

$$S(t_+, t_-) = e^{i\hat{H}_0 t_+} e^{-i\hat{H}(t_+ - t_-)} e^{-i\hat{H}_0 t_-} \quad (4.11)$$

which is unitary

$$S(t_+, t_-)^\dagger S(t_+, t_-) = S(t_+, t_-) S(t_+, t_-)^\dagger = 1 \quad (4.12)$$

where the total Hamiltonian $\hat{H} = \int d^3x \mathcal{H}$ is defined by

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} \Pi^2(0, \vec{x}) + \frac{1}{2} \vec{\nabla} \phi(0, \vec{x}) \vec{\nabla} \phi(0, \vec{x}) + \frac{1}{2} m^2 \phi^2(0, \vec{x}) \\ &+ \frac{g}{2 \cdot 3!} [\phi(0, \vec{x}) \star \phi(0, \vec{x}) \star \phi(0, \vec{x}) + h.c.] \end{aligned} \quad (4.13)$$

with the naive canonical momentum $\Pi(x) = \frac{\partial}{\partial x^0} \phi(x)$ conjugate to the variable $\phi(x)$. See [15] for a different approach to the Hamiltonian formulation of space-time noncommutative theory.

The modified time ordering for the non-local Hamiltonian (4.13), which defines a unitary S-matrix (4.11) in a perturbative expansion, however spoils the positive energy condition [8] in the sense that only the positive energy flows in the positive time direction for any fixed time-slice in space-time, just as (3.25) in the naive theory non-local in time in the previous section.

5 Conclusion

We discussed some of the basic aspects of quantized theory which is non-local in the time variable on the basis of path integral quantization. We analyzed the recent proposal of the modified time ordering prescription[5, 6], which generally defines a unitary S-matrix for theories non-local in time. It has been shown that the unitary S-matrix has certain advantages but at the time it has several disadvantages, and the perturbative positive energy condition and Wick rotation to Euclidean theory, which are ensured by the Feynman's $m^2 - i\epsilon$ prescription in the path integral, are spoiled. The modified time ordering needs to be examined further whether it is consistent with other basic postulates in quantized field theory.

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